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Similarity reductions for variable-coefficient coupled nonlinear Schrödinger equations

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Abstract. We categorize classes of coupled nonlinear Schrödinger equations which allow generalized similarity solutions, using the approach of Clarkson and Kruskal (1989). In all cases, the resulting pair of ordinary differential equations belongs to a single class, presented here as equations (2.6). Certain cases allowing solution in terms of familiar functions are identified. An alternative approach, presented in section 4, shows that only two conditions need be placed on the four real and two complex coefficients in the governing equations in order that solutions generated by an arbitrary solution of the (integrable) NLS equation exist. Applications to some standard coupled systems arising from fibre optics are given.

1. Introduction

Many descriptions of two nonlinearly coupled modulated wavetrains, particularly in fibre optics, lead to a coupled pair of nonlinear Schrödinger (CNLS) equations. Although the single nonlinear Schrödinger (NLS) equation with constant coefficients

$$\mathrm{i}u_t = u_{xx} + |u|^2 u \tag{1.1}$$

is completely integrable by the inverse scattering transform (Zakharov and Shabat 1972), CNLS systems with constant coefficients are found to be completely integrable (Zakharov and Schulman 1982) only in the case

$$\mathrm{i}u_t = u_{xx} + (|u|^2 + |v|^2)u \qquad \mathrm{i}v_t = v_{xx} + (|u|^2 + |v|^2)v \tag{1.2}$$

for which Manakov (1973) found explicit soliton solutions.

As the technology of optical fibres for long distance communication and signal processing has rapidly developed, a large variety of CNLS systems have arisen and been investigated analytically and numerically. For example, constant coefficient pairs of equations of the class

$$\mathrm{i}A^{\pm}_{\tau} + A^{\pm}_{ss} \pm \Delta A^{\pm} + \kappa A^{\mp} + (|A^{\pm}|^2 + h|A^{\mp}|^2)A^{\pm} = 0 \tag{1.3}$$

where $A^{+}(s, \tau)$ and $A^{-}(s, \tau)$ are two complex mode-amplitudes in a birefringent fibre while Δ, κ and h are real constants, have been treated numerically by Trillo *et al* (1989). They include as special cases the systems (i) $\Delta = 0$, $h = 0$ describing directional couplers (Trillo *et al* 1988, Kivshar and Malomed 1989) and (ii) $\Delta = 0$ for birefringent fibres, for which Florjanczyk and Tremblay (1989) and Kostov and Uzunov (1992) have identified various families of solutions involving a factor $\exp(\mathrm{i}\Omega\tau)$ multiplying Jacobian elliptic functions of s. This case (ii) also corresponds to two other classes of known exact solutions. The pair of equations

$$\begin{aligned} &\mathrm{i}u_\tau + u_{ss} + (|u|^2 + \sigma|v|^2)u + (1-\sigma)v^2u^* \,\mathrm{e}^{-2\mathrm{i}\kappa\tau} = 0 \\ &\mathrm{i}v_\tau + v_{ss} + (|v|^2 + \sigma|u|^2)v + (1-\sigma)u^2v^* \,\mathrm{e}^{2\mathrm{i}\kappa\tau} = 0 \end{aligned} \tag{1.4}$$

is reducible to the system (1.3) with $\Delta = 0$, $h = 2\sigma^{-1} - 1$ by the substitutions $u = (2\sigma)^{-1/2}(A^+ + A^-)\exp(-\mathrm{i}\kappa\tau)$, $v = \mathrm{i}(2\sigma)^{-1/2}(A^+ - A^-)\exp(\mathrm{i}\kappa\tau)$, while the pair of equations

$$\mathrm{i}e^\pm_t + e^\pm_{xx} \pm \kappa e^\pm + a(|e^+|^2 + |e^-|^2)e^\pm + b\{(e^+)^2 + (e^-)^2\}e^{\pm *} = 0 \tag{1.5}$$

may be reduced to (1.3) with $\Delta = 0$, $h = 1 + 2b/a$ by the substitutions $e^+ = (2a)^{-1/2}(A^+ + A^-)$, $e^- = \mathrm{i}(2a)^{-1/2}(A^- - A^+)$, $\tau = t$, $s = x$. Here, and henceforth, * denotes a complex conjugate. For (1.4), Christodoulides and Joseph (1988) showed that stationary (frozen) states, i.e. waveforms with envelope depending only on s, satisfy an integrable Hamiltonian system of ordinary differential equations. They determine some novel 'vector soliton' solutions of these in terms of hyperbolic functions. For (1.5), Tratnik and Sipe (1988) determine formulae describing some more general polarization-modulated stationary states.

Other sets of equations for which explicit exact solutions have been determined include

$$\mathrm{i}(u^\pm_\tau \pm \delta u^\pm_s) \pm \Delta u^\pm + u^\pm_{ss} + (|u^\pm|^2 + h|u^\mp|^2)u^\pm = 0 \tag{1.6}$$

for which Tratnik (1992) has found single and multiple 'twisted soliton' solutions which may exist due to non-vanishing group delay difference ($\delta \neq 0$)

$$\mathrm{i}(u^\pm_\tau \pm \delta u^\pm_s) + \kappa u^\mp + (|u^\pm|^2 + h|u^\mp|^2)u^\pm = 0 \tag{1.7}$$

for which Aceves and Wabnitz (1989) have obtained 'Bragg solitons' describing the self-trapping of two counter-propagating beams in a periodic nonlinear medium. Additionally, various related systems have been investigated numerically:

$$\mathrm{i}(A^\pm_\tau \pm \delta A^\pm_s) + A^\pm_{ss} \pm \Delta A^\pm + \kappa A^\mp + (|A^\pm|^2 + h|A^\mp|^2)A^\pm = 0 \tag{1.8}$$

(Wabnitz *et al* 1990) which introduces into (1.3) the group delay difference δ:

$$\mathrm{i}A^\pm_\tau + \beta^\pm A^\pm_{ss} + R^\pm(|A^\pm|^2 + 2|A^\mp|^2)A^\pm = 0 \tag{1.9}$$

in which Trillo *et al* (1988) investigate the stability of an exact solution in which a hyperbolic secant for A^- in the normal dispersion ($\beta^- < 0$) mode is coupled through cross-phase modulation to a hyperbolic tangent for A^+ in the anomolous dispersion ($\beta^+ > 0$) mode:

$$\mathrm{i}(A^\pm_\tau \pm \delta A^\pm_s) + \kappa A^\mp \exp(\mp 2\mathrm{i}\nu\tau) + A^\pm_{ss} + (|A^\pm|^2 + h|A^\mp|^2)A^\pm = 0 \tag{1.10}$$

derived for periodically twisted birefringent fibres by Wabnitz *et al* (1991) and further investigated by Aceves and Wabnitz (1992):

$$\begin{aligned} &\mathrm{i}(u^\pm_\tau \pm \delta u^\pm_s) - \kappa^\pm(\tau)u^\mp \exp(\mp 2\mathrm{i}\nu\tau) + u^\pm_{ss} + (|u^\pm|^2 + \sigma|u^\mp|^2)u^\pm \\ &\qquad + (1-\sigma)(u^\mp)^2 u^{*\mp} \exp(\mp 4\mathrm{i}\nu\tau) = 0 \end{aligned} \tag{1.11}$$

with $\sigma = \frac{2}{3}$ and $\kappa^+(\tau) = [\kappa^-(\tau)]^*$ a random coupling coefficient, which De Angelis *et al* (1992) use to describe random coupling between amplitudes $u^\pm$ of linearly polarized modes with a group delay difference ($\delta \neq 0$).

Other systems with coefficients depending on t (or τ), the distance along the fibre, arise naturally in long-distance communications where amplification occurs periodically at spacing short compared to a soliton evolution distance. Under these conditions, effective constant-coefficient systems are obtained using the guiding-centre description (essentially a multiple scales formulation). However, this situation, like that of directional couplers and pumped fibres (Abdullaev *et al* 1989), motivates the need to classify CNLS systems with variable coefficients for which classes of explicit solution can be found. In this paper, we confine attention to systems

$$\mathrm{i}\tilde{u}_t^{\pm} = \tilde{u}_{xx}^{\pm} + \tilde{\gamma}_{\pm}(t)\tilde{u}^{\pm} + \tilde{\varepsilon}_{\pm}(t)\tilde{u}^{\mp} + \{\tilde{\alpha}_{\pm}(t)|\tilde{u}^{+}|^2 + \tilde{\beta}_{\pm}(t)|\tilde{u}^{-}|^2\}\tilde{u}^{\pm} \tag{1.12}$$

with $\tilde{\alpha}_{\pm}, \tilde{\beta}_{\pm} \in \mathbb{R}$, $\tilde{\gamma}_{\pm}, \tilde{\varepsilon}_{\pm} \in \mathbb{C}$, which generalize the class (1.3), since the system (1.6) like equations (1.3)-(1.5) can be transformed into this form (see appendix).

2. Similarity reductions and some special solutions

This section is devoted to the similarity analysis of the general pair of coupled NLS equations

$$\mathrm{i}u_t^{\pm} = u_{xx}^{\pm} + \{\alpha_{\pm}(t)|u^{+}|^2 + \beta_{\pm}(t)|u^{-}|^2\}u^{\pm} + \varepsilon_{\pm}(t)u^{\mp} \tag{2.1}$$

with $\alpha_{\pm}, \beta_{\pm} \in \mathbb{R}$, $\varepsilon_{\pm} \in \mathbb{C}$, obtained from the canonical system (1.12) by the transformations

$$\begin{aligned}
&\tilde{u}^{\pm}(x,t) = P_{\pm}(t)\exp(-\mathrm{i}\Gamma_{\pm})u^{\pm}(x,t) \qquad \tilde{\gamma}_{\pm} = \Gamma_{\pm}'(t) + \mathrm{i}\frac{P_{\pm}'(t)}{P_{\pm}} \\
&\alpha_{\pm} = P_{+}^2\tilde{\alpha}_{\pm} \qquad \beta_{\pm} = P_{-}^2\tilde{\beta}_{\pm} \qquad \varepsilon_{\pm} = (P_{\mp}/P_{\pm})\tilde{\varepsilon}_{\pm}\exp(\mathrm{i}(\Gamma_{\pm} - \Gamma_{\mp}))
\end{aligned} \tag{2.2}$$

where here the henceforth primes denote ordinary differentiation.

Following the recent approach proposed by Clarkson and Kruskal (1989), we search for similarity solutions of (2.1) in the general form

$$u^{\pm}(x,t) = U^{\pm}(x,t,v_{\pm}(z)) \qquad z = z(x,t) \tag{2.3}$$

with the requirement that substitution of (2.3) into (2.1) should yield a pair of ordinary differential equations for $v_{\pm}(z)$. This process determines the functional forms of $U^{\pm}$ and of $z(x,t)$, as well as imposing functional restrictions on the coefficients entering (2.1). In the present case, it is easily ascertained by direct substitution into (2.1) that (2.3) specializes to

$$u^{\pm}(x,t) = m_{\pm}(t)f_{\pm}(z)\exp[\mathrm{i}\varphi^{\pm}(x,t)] \tag{2.4}$$

where

$$z = x\theta(t) + \psi(t). \tag{2.5}$$

In (2.4) and (2.5), $m_{\pm}$, $\varphi^{\pm}$, θ and ψ are real functions to be determined, while $f_{\pm} \in \mathbb{C}$.

By insisting that x and t do not arise explicitly in the ordinary differential equations governing $f_{\pm}(z)$, it is found after considerable manipulation that six cases arise. In each, the resulting differential equations can be put into the form

$$f_{\pm}''(z) + (\mu + \nu z + \lambda z^2 + \mathrm{i}\kappa_{\pm})f_{\pm} + \{\bar{\alpha}_{\pm}|f_{+}|^2 + \bar{\beta}_{\pm}|f_{-}|^2\}f_{\pm} + \bar{\varepsilon}_{\pm}f_{\mp} = 0 \tag{2.6}$$

where μ, ν, λ, $\kappa_{\pm}$, $\bar{\alpha}_{\pm}$ and $\bar{\beta}_{\pm}$ are real constants, $\bar{\varepsilon}_{\pm}$ are complex constants and primes denote differentiation. The six cases are:

1. Travelling waves. $z = x - 2Vt$, $m_\pm = \exp(-\kappa_\pm t)$, $\varphi^\pm = \varphi(x,t) = -Vx + (V^2 + \mu)t$ (V = constant). These solutions exist provided that

$$\alpha_\pm = \bar{\alpha}_\pm \, e^{2\kappa_+ t} \qquad \beta_\pm = \bar{\beta}_\pm \, e^{2\kappa_- t} \qquad \varepsilon_\pm = \bar{\varepsilon}_\pm \, e^{\pm(\kappa_- - \kappa_+)} \tag{2.7}$$

while the coefficients in (2.6) are specialized so that $\nu = 0$, $\lambda = 0$, thus making the system become autonomous.

2. Accelerating waves. $z = x + \nu t^2$, $m_\pm = \exp(-\kappa_\pm t)$, $\varphi^\pm = \varphi(x,t) = \nu x t + \frac{2}{3}\nu^2 t^3 + \mu t$, with the coefficients in (2.1) taking the same forms (2.7) as in case 1. In the ODEs (2.6) we have $\nu \neq 0$, $\lambda = 0$, so that the system becomes a coupled pair of Painlevé II type.

3. $z = (x + Vt)t^{-1/2}$, $m_\pm = t^{-(\kappa_\pm + \frac{1}{4})}$ (V = constant),

$$\varphi^\pm = \varphi(x,t) = \tfrac{1}{8} t^{-1}(V^2 t^2 + 2Vxt - x^2) + \mu \ln t.$$

These solutions exist provided that

$$\alpha_\pm = \bar{\alpha}_\pm t^{2\kappa_+ - \frac{1}{2}} \qquad \beta_\pm = \bar{\beta}_\pm t^{2\kappa_- - \frac{1}{2}} \qquad \varepsilon_\pm = \bar{\varepsilon}_\pm t^{\kappa_\mp - \kappa_\pm - 1}. \tag{2.8}$$

The corresponding choice of coefficients in (2.6) is $\nu = 0$, $\lambda = \frac{1}{16}$.

4. $z = xt^{-1} + \nu t^{-2}$, $m_\pm = t^{-\frac{1}{2}} \exp(\kappa_\pm t^{-1})$,

$$\varphi^\pm = \varphi(x,t) = -\tfrac{1}{4} x^2 t^{-1} - \nu x t^{-2} - \tfrac{2}{3}\nu^2 t^{-3} + \mu t^{-1}.$$

These solutions exist provided that

$$\alpha_\pm = \bar{\alpha}_\pm t^{-1} \, e^{-2\kappa_+ t^{-1}} \qquad \beta_\pm = \bar{\beta}_\pm t^{-1} \, e^{-2\kappa_- t^{-1}} \qquad \varepsilon_\pm = \bar{\varepsilon}_\pm t^{-1} \, e^{\pm(\kappa_+ - \kappa_-)t} \tag{2.9}$$

with $\lambda = 0$ in (2.6).

5. $z = (x + Vt)(t^2 + \delta^2)^{-\frac{1}{2}}$, $m_\pm = (t^2 + \delta^2)^{-\frac{1}{4}} \exp[(-\kappa_\pm/\delta)\tan^{-1}(t/\delta)]$, $\varphi^\pm = \varphi(x,t) = \frac{1}{4}(\delta^2 V^2 t + 2\delta^2 V x - t x^2)(t^2 + \delta^2)^{-1} + (\mu/\delta)\tan^{-1}(t/\delta)$ (V, δ are constants). These solutions exist provided that

$$\begin{aligned} \alpha_\pm &= \bar{\alpha}_\pm (t^2 + \delta^2)^{-\frac{1}{2}} \exp[(2\kappa_+/\delta)\tan^{-1}(t/\delta)] \\ \beta_\pm &= \bar{\beta}_\pm (t^2 + \delta^2)^{-\frac{1}{2}} \exp[(2\kappa_-/\delta)\tan^{-1}(t/\delta)] \\ \varepsilon_\pm &= \bar{\varepsilon}_\pm (t^2 + \delta^2)^{-1} \exp[\delta^{-1}(\kappa_\mp - \kappa_\pm)\tan^{-1}(t/\delta)] \end{aligned} \tag{2.10}$$

and yield coefficients in (2.6) of the form $\nu = 0$, $\lambda = -\frac{1}{4}\delta^2$.

6. $z = (x + Vt)(t^2 - \delta^2)^{-\frac{1}{2}}$, $m_\pm = (t^2 - \delta^2)^{-\frac{1}{4}}[(t+\delta)/(t-\delta)]^{\kappa_\pm/2\delta}$, $\varphi^\pm = \varphi(x,t) = -\frac{1}{4}(\delta^2 V^2 t + 2\delta^2 V x + t x^2)(t^2 - \delta^2)^{-1} + (\mu/2\delta)\ln|(t-\delta)/(t+\delta)|$ (V, δ are constants). These solutions exist provided that

$$\begin{aligned} \alpha_\pm &= \bar{\alpha}_\pm (t^2 - \delta^2)^{-\frac{1}{2}} ((t-\delta)/(t+\delta))^{\kappa_+/\delta} \\ \beta_\pm &= \bar{\beta}_\pm (t^2 - \delta^2)^{-\frac{1}{2}} ((t-\delta)/(t+\delta))^{\kappa_-/\delta} \\ \varepsilon_\pm &= \bar{\varepsilon}_\pm (t^2 - \delta^2)^{-\frac{1}{2}} ((t-\delta)/(t+\delta))^{\pm(\kappa_- - \kappa_+)/2\delta}. \end{aligned} \tag{2.11}$$

The corresponding set of coefficients in (2.6) is $\nu = 0$, $\lambda = \frac{1}{4}\delta^2$.

The similarity variables z arising in Cases 1–3 are all included amongst those found in Parker (1988) by Lie group analysis for equations (2.1) with $\varepsilon_\pm = 0$ and with $\alpha_\pm$, $\beta_\pm$ all constant. This, together with the forms found for $\alpha_\pm$, $\beta_\pm$ and $\varepsilon_\pm$ in each case, suggests that existence of similarity solutions (2.4) is closely allied to the ability to reduce coefficients in (2.2) to constants. The simplifications found are as follows:

Cases 1, 2: $u^\pm = m_\pm(t)\exp(\mathrm{i}\mu t) v^\pm(x,t)$

$$\mathrm{i} v_t^\pm = v_{xx}^\pm + (\bar{\alpha}_\pm |v^+|^2 + \bar{\beta}_\pm |v^-|^2) v^\pm + (\mathrm{i}\kappa_\pm + \mu) v^\pm + \bar{\varepsilon}_\pm v^\mp \tag{2.12}$$

Case 3: $u^{\pm}=t^{\frac{1}{2}}m_{\pm}(t)\exp(\mathrm{i}\mu\ln t)v^{\pm}(x,t)$

$$\mathrm{i}v_t^{\pm}=v_{xx}^{\pm}+(\bar{\alpha}_{\pm}|v^{+}|^2+\bar{\beta}_{\pm}|v^{-}|^2)v^{\pm}+t^{-1}\{(\mathrm{i}\kappa_{\pm}+\mu)-\tfrac{1}{4}\mathrm{i}\}v^{\pm}+t^{-1}\bar{\varepsilon}_{\pm}v^{\mp} \tag{2.13}$$

Cases 4, 5 and 6: $u^{\pm}=(t^2+D)^{\frac{1}{2}}m_{\pm}(t)\exp(\mathrm{i}\Gamma(t))v^{\pm}(x,t)$

$$\mathrm{i}v_t^{\pm}=v_{xx}^{\pm}+(\bar{\alpha}_{\pm}|v^{+}|^2+\bar{\beta}_{\pm}|v^{-}|^2)v^{\pm}+(t^2+D)^{-1}\{(\mathrm{i}\kappa_{\pm}+\mu-\tfrac{1}{2}\mathrm{i}t)v^{\pm}+\bar{\varepsilon}_{\pm}v^{\mp}\} \tag{2.14}$$

with $D=0$ and $\Gamma(t)=\mu t^{-1}$ (Case 4), $D=\delta^2$ and $\Gamma(t)=(\mu/\delta)\tan^{-1}(t/\delta)$ (Case 5), $D=-\delta^2$ and $\Gamma(t)=(\mu/\delta)\ln|(t-\delta)/(t+\delta)|$ (Case 6). In these three cases the respective forms of $(t^2+D)^{\frac{1}{2}}m_{\pm}(t)$ are

$$tm_{\pm}(t)=t^{\frac{1}{2}}\exp(\kappa_{\pm}t^{-1}) \qquad \text{(Case 4)}$$

$$(t^2+\delta^2)^{\frac{1}{2}}m_{\pm}(t)=(t^2+\delta^2)^{\frac{1}{4}}\exp\{-\delta^{-1}\kappa_{\pm}\tan^{-1}(t/\delta)\} \qquad \text{(Case 5)}$$

$$(t^2+\delta^2)^{\frac{1}{2}}m_{\pm}(t)=(t^2-\delta^2)^{\frac{1}{4}}\left(\frac{t+\delta}{t-\delta}\right)^{\kappa_{\pm}/2\delta} \qquad \text{(Case 6)}$$

The above reductions show that, for a pair of equations (2.1) with constant coefficients but non-vanishing linear undifferentiated terms, the only permissible similarity solutions are the travelling waves and accelerating waves. More generally, we conclude that similarity solutions cannot arise for (2.1), unless that system can be reduced to one of the forms (2.13) or (2.14) with constant coefficients in the nonlinear and differentiated terms and with the remaining linear terms having either constant coefficients, coefficients proportional to t^{-1} or with multipliers $(t^2+D)^{-1}$, $t(t^2+D)^{-1}$.

Some exact solutions

To conclude this section we describe certain closed-form solutions to (2.6) which may be obtained in special cases. To achieve this, we seek particular solutions in the form

$$f_+(z)=p(z)\,\mathrm{e}^{\mathrm{i}q(z)} \qquad f_-(z)=ap(z)\,\mathrm{e}^{\mathrm{i}cq(z)} \tag{2.15}$$

where $p(z)$ and $q(z)$ are real functions, while a and c are real constants.

Substituting (2.15) into (2.6), separating into real and imaginary parts, writing $\tilde{\varepsilon}_{\pm}=\operatorname{Re}\bar{\varepsilon}_{\pm}$, $\hat{\varepsilon}_{\pm}=\operatorname{Im}\bar{\varepsilon}_{\pm}$ and requiring that the multipliers of the various types of term trigonometric in $q(z)$ vanish individually leads to just the two possibilities $c=\pm1$, in each of which $a^2(\bar{\beta}_+-\bar{\beta}_-)+\bar{\alpha}_+-\bar{\alpha}_-=0$. The other conditions in these cases are

1. $c=1$, requiring $\mathrm{i}(a\bar{\varepsilon}_+ + a^{-1}\bar{\varepsilon}_-)=\kappa_+-\kappa_-$ or equivalently

$$a^2\hat{\varepsilon}_+-\hat{\varepsilon}_-+a(\kappa_+-\kappa_-)=0 \qquad \text{with } a^2\tilde{\varepsilon}_+=\tilde{\varepsilon}_-. \tag{2.16}$$

Here $f_-(z)=af_+(z)$, with $f_+(z)$ governed by

$$f_+''(z)+\{\bar{\mu}+\nu z+\lambda z^2+\mathrm{i}\delta+\Lambda|f_+|^2\}f_+=0 \tag{2.17}$$

where $\bar{\mu}=\mu+(\tilde{\varepsilon}_+\tilde{\varepsilon}_-)^{\frac{1}{2}}$, $\delta=\kappa_++a\hat{\varepsilon}_+$, $\Lambda=\bar{\alpha}_++\tilde{\varepsilon}_-\beta_+/\tilde{\varepsilon}_+$.

2. $c=-1$, requiring $a\bar{\varepsilon}_+-a^{-1}\bar{\varepsilon}_-^*=0$ and $\kappa_+=-\kappa_-$ or equivalently

$$a^2\hat{\varepsilon}_++\hat{\varepsilon}_-=0 \qquad a^2\tilde{\varepsilon}_+=\tilde{\varepsilon}_- \qquad \text{with } \kappa_+=-\kappa_-. \tag{2.18}$$

In this case, $f_-(z)=af_+^*(z)$ with $f_+(z)$ governed by

$$f_+''(z)+\{\mu+\nu z+\lambda z^2+\mathrm{i}\kappa_++\Lambda|f_+|^2\}f_++a\bar{\varepsilon}_+f_+^*=0 \tag{2.19}$$

with * denoting a complex conjugate.

In terms of $p(z)$ and $q(z)$, equation (2.17) becomes

$$p''(z)-p(q')^2+(\bar{\mu}+\nu z+\lambda z^2)p+\Lambda p^3=0 \qquad 2p'q'+pq''+\delta p=0 \tag{2.20}$$

while, for $c=-1$, equation (2.19) gives a system generalizing (2.20), with

$$\delta=\kappa_+ + a(\hat{\varepsilon}_+\cos 2q-\tilde{\varepsilon}_+\sin 2q)\equiv\delta(z)$$

$$\bar{\mu}=\mu+a(\tilde{\varepsilon}_+\cos 2q+\hat{\varepsilon}_+\sin 2q)\equiv\bar{\mu}(z).$$

Thus, if the real constants $\bar{\alpha}_\pm$, $\bar{\beta}_\pm$, $\hat{\varepsilon}_\pm$, $\tilde{\varepsilon}_\pm$ and $\kappa_\pm$ with $a=\pm(\tilde{\varepsilon}_-/\tilde{\varepsilon}_+)^{\frac{1}{2}}$ and $c=\pm 1$ satisfy $\tilde{\varepsilon}_-(\bar{\beta}_+-\bar{\beta}_-)+\tilde{\varepsilon}_+(\bar{\alpha}_+-\bar{\alpha}_-)=0$ and (2.16) or (2.18), then to any solution of the system (2.20) there corresponds, through (2.15), a solution of (2.6).

In the following, we confine attention to case 1 with $\kappa_+=-a\hat{\varepsilon}_+$ ($\delta=0$), so obtaining from (2.20)

$$q'(z)=q_0p^{-2} \qquad p''(z)+(\bar{\mu}+\nu z+\lambda z^2)p+\Lambda p^3-q_0^2p^{-3}=0 \tag{2.21}$$

where q_0 is an arbitrary constant. We then identify the following special cases:

(i) If $\lambda=q_0=0$ and $\Lambda<0$, the transformation

$$p=\nu^{\frac{1}{3}}(-2/\Lambda)^{\frac{1}{2}}v(\tilde{z}) \qquad \tilde{z}=-\nu^{-\frac{2}{3}}(\bar{\mu}+\nu z) \tag{2.22}$$

reduces (2.21), to Painlevé II (see Ince, 1956)

$$v''(\tilde{z})=2v^3+\tilde{z}v. \tag{2.23}$$

This possibility arises for Cases 2 and 4 and allows the possibility of asymmetric localized pulses with algebraic decay as $\tilde{z}\to-\infty$ (see Ablowitz and Clarkson, 1991).

(ii) If $\lambda=q_0=\Lambda=0$, setting $\tilde{z}=-(z+\bar{\mu}/\nu)$ reduces (2.21) to the Airy equation

$$p''(\tilde{z})-\tilde{z}p(\tilde{z})=0 \tag{2.24}$$

which can be integrated in terms of Bessel sunctions of order $\frac{1}{3}$ (see Whittaker and Watson 1927). These solutions, in terms of solutions of the linear equations (2.24), arise only when $\bar{\alpha}_+/\bar{\beta}_+=\bar{\alpha}_-/\bar{\beta}_-=-\tilde{\varepsilon}_-/\tilde{\varepsilon}_+<0$. They require that the terms in braces in (2.6) simultaneously vanish identically. The solutions can apply only to Cases 2 and 4.

(iii) If $\nu=\lambda=0$, equation (2.21) becomes autonomous and may be integrated to give

$$(p')^2+\bar{\mu}p^2+\tfrac{1}{2}\Lambda p^4+q_0^2p^{-2}=q_1=\text{constant}. \tag{2.25}$$

This may be reduced for $\Lambda\neq 0$, by using the transformation

$$v(\tilde{z})=-(p^2+\tfrac{2}{3}\Lambda^{-1}\bar{\mu})\operatorname{sgn}\Lambda \qquad \tilde{z}=(\tfrac{1}{2}|\Lambda|)^{\frac{1}{2}}z$$

to

$$v'(\tilde{z})=\pm(4v^3-q_2v-q_3)^{\frac{1}{2}} \tag{2.26}$$

where

$$q_2=8\Lambda^{-2}\{\tfrac{2}{3}\bar{\mu}^2+q_1\Lambda\} \qquad q_3=8\Lambda^{-3}\{\tfrac{8}{27}\bar{\mu}^3+q_0^2\Lambda^2+\tfrac{2}{3}q_1\bar{\mu}\}\operatorname{sgn}\Lambda.$$

Equation (2.26) may be solved in terms of elliptic functions or, more, generally, the Weierstrass P-function (see Whittaker and Watson 1927).

As a particular case we find that, for $q_0=q_1=0$, $\bar{\mu}<0$ and $\Lambda>0$, the general solution of (2.25) has the form

$$p(z)=\Lambda_0\operatorname{sech}[\Lambda_0(z-z_0)] \qquad \Lambda_0^2=-2\bar{\mu}/\Lambda \tag{2.27}$$

where z_0 is an arbitrary constant and the similarity variable is $z = x - 2Vt$ (Case 1) or $z = x/t$ (Case 4).

Additionally, for the exceptional situation $\bar{\alpha}_+/\bar{\beta}_+ = \bar{\alpha}_-/\bar{\beta}_- = -\tilde{\varepsilon}_-/\tilde{\varepsilon}_+ < 0$ giving $\Lambda = 0$ as in (ii), the solution to (2.25) is found as

$$p^2 = \{q_1 + (q_1^2 - \bar{\mu} q_0^2)^{\frac{1}{2}} \cos(2\bar{\mu}^{\frac{1}{2}} z)\}/2\bar{\mu}$$

$$q = 2\tan^{-1}\{M \tan(\bar{\mu}^{\frac{1}{2}} z)\} \qquad M^2 = \frac{q_1 - (q_1^2 - \bar{\mu} q_0^2)^{\frac{1}{2}}}{q_1 + (q_1^2 - \bar{\mu} q_0^2)^{\frac{1}{2}}}$$

omitting a trivial translation in z.

(iv) If $\Lambda = q_0 = 0$, $\lambda < 0$, equation (2.21) reduces to

$$p''(\hat{z}) - (\tfrac{1}{4}\hat{z}^2 - \hat{\mu})p = 0 \tag{2.28}$$

where $\hat{z} = (-4\lambda)^{\frac{1}{4}}(z + \frac{1}{2}\nu\lambda^{-1})$, $\hat{\mu} = \frac{1}{8}(-\lambda)^{\frac{1}{2}}(4\lambda\bar{\mu} - \nu^2)$. Equation (2.28) is Weber's equation which can be solved in terms of parabolic cylinder Whittaker functions. This possibility arises only for $\bar{\alpha}_+/\bar{\beta}_+ = \bar{\alpha}_-/\bar{\beta}_- = -\tilde{\varepsilon}_-/\tilde{\varepsilon}_+ < 0$ with Case 5. An equivalent equation for $\lambda > 0$ can arise for Cases 3 and 6.

3. Solutions with amplitude depending only on t

The previous section concerns reductions in terms of a similarity variable $z(x, t)$ for which $z_x \neq 0$. In this section our aim is to consider the degenerate case $z_x = 0$, seeking particular solutions of (2.1) in the form

$$u^{\pm}(x, t) = f_{\pm}(t) \exp i\varphi^{\pm}(x, t) \tag{3.1}$$

in which $f_{\pm}$ and $\varphi^{\pm}$ are real. In these solutions, $|u^{\pm}| = f_{\pm}(t)$ depends only on the evolution coordinate t.

Substituting (3.1) into (2.1) and separating the real and imaginary parts yields the following four real differential equations involving the functions $f_{\pm}$ and $\varphi^{\pm}$ as well as the coefficients $\alpha_{\pm}(t)$, $\beta_{\pm}(t)$ and $\varepsilon_{\pm}(t) = \rho_{\pm}(t)\exp\{i\sigma_{\pm}(t)\}$, $(\rho_{\pm}, \sigma_{\pm} \in \mathbb{R})$:

$$\begin{aligned} &f_{\pm}\varphi^{\pm}_{xx} - f'_{\pm} + \rho_{\pm} f_{\mp} \sin(\varphi^{\mp} - \varphi^{\pm} + \sigma_{\pm}) = 0 \\ &f_{\pm}\{\varphi^{\pm}_t - (\varphi^{\pm}_x)^2 + \alpha_{\pm} f_+^2 + \beta_{\pm} f_-^2\} + \rho_{\pm} f_{\mp} \cos(\varphi^{\mp} - \varphi^{\pm} + \sigma_{\pm}) = 0. \end{aligned} \tag{3.2}$$

First we consider the case $\varepsilon_{\pm} = 0$, for which equations (3.2) can be integrated to give, through (3.1), the solutions

$$u^{\pm}(x, t) = m_0^{\pm}(t - t_{\pm})^{-\frac{1}{2}} \exp i\left\{\frac{-(x - b_{\pm})^2}{4(t - t_{\pm})} + R_{\pm}(t)\right\} \tag{3.3}$$

where $R_{\pm}$ are given by integration of

$$R'_{\pm}(t) = -m_0^{+}\frac{\alpha_{\pm}(t)}{t - t_+} - m_0^{-}\frac{\beta_{\pm}(t)}{t - t_-} \tag{3.4}$$

and $m_0^{\pm}$, $t_{\pm}$, $b_{\pm}$ are arbitrary real constants. It may be observed that the solutions (3.3) to (2.1) exist for every choice of functions $\alpha_{\pm}(t)$, $\beta_{\pm}(t)$, which affect only the local 'frequency' $\varphi^{\pm}_t$. In particular, it may be noted that solutions (3.3) apply to the system (1.6), since it is shown in the appendix that (1.6) may be reduced to the constant coefficient system (A.5). Moreover, (A.5) is the system for which Parker (1988) first found a special case of solutions (3.3) in which $t_- = t_+$.

We now consider $\varepsilon_+ \neq 0$, $\varepsilon_- \neq 0$ and investigate the possibilities for which $\varphi^{\mp} - \varphi^{\pm} + \sigma_\pm = d_\pm$ are each constants. These require that the arguments $\sigma_\pm$ of $\varepsilon_\pm$ are related by $\sigma_+(t) + \sigma_-(t) = d_+ + d_- =$ constant. We write

$$\sigma_\pm = \tilde{a} \pm \tilde{b}(t) \qquad \varphi^+ - \varphi^- = \tilde{b}(t) + \tfrac{1}{2}(d_- - d_+) \qquad \tilde{a} = \tfrac{1}{2}(d_+ + d_-). \tag{3.5}$$

This gives, after substitution into (3.2) followed by considerable manipulation, the following solutions to (2.1):

$$\begin{aligned} u^\pm(x,t) &= \frac{\hat{f}_\pm(t)}{\sqrt{t-t_0}} \exp \mathrm{i}\left\{ \frac{-x^2}{4(t-t_0)} - \psi(t) \pm \tfrac{1}{2}\tilde{b}(t) - \tfrac{1}{2}d_\pm \right\} \\ 2\psi'(t) &= (t-t_0)^{-1}\{(\alpha_+ + \alpha_-)\hat{f}_+^2 + (\beta_+ + \beta_-)\hat{f}_-^2\} + \rho_+ \cos d_+ \frac{\hat{f}_-}{\hat{f}_+} + \rho_- \cos d_- \frac{\hat{f}_+}{\hat{f}_-} \end{aligned} \tag{3.6}$$

along with the conditions (3.5) on $\sigma_+ + \sigma_-$ and

$$(t-t_0)^{-1}\{(\alpha_+ - \alpha_-)\hat{f}_+^2 + (\beta_+ - \beta_-)\hat{f}_-^2\} + \rho_+ \cos d_+ \frac{\hat{f}_-}{\hat{f}_+} - \rho_- \cos d_- \frac{\hat{f}_+}{\hat{f}_-} + \tilde{b}'(t) = 0. \tag{3.7}$$

In this solution, and in the constraint (3.7), the functions $\hat{f}_\pm(t)$ are solutions of the ordinary differential equations

$$\hat{f}'_\pm(t) = (\rho_\pm(t) \sin d_\pm)\hat{f}_\mp(t). \tag{3.8}$$

Thus, given the expressions $\rho_\pm(t)$, if we integrate equations (3.8) and ensure that $\alpha_\pm(t)$, $\beta_\pm(t)$ are consistent with (3.7) while $\sigma_+(t) + \sigma_-(t) = d_+ + d_-$, then the solution (3.6) satisfies (2.1).

Alternatively, by assigning the functions $\hat{f}_\pm(t)$, we may determine classes of coefficients $\alpha_\pm(t)$, $\beta_\pm(t)$, $\rho_\pm(t)$, $\tilde{b}(t)$ and constants $d_\pm$, t_0 which allow physically interesting solutions to (2.1) of the form (3.6).

3.1. Bound-state solutions

It is possible to find another class of solutions of the form (3.1) by following a different strategy, based on the fact that complex solutions $F_\pm(t)$ of the coupled differential equations

$$\begin{aligned} \mathrm{i}(V+1)F'_+(t) &= \lambda F_+ + F_- + (|F_+|^2 + h|F_-|^2)F_+ \\ \mathrm{i}(V-1)F'_-(t) &= F_+ + \lambda F_- + (h|F_+|^2 + |F_-|^2)F_- \end{aligned} \tag{3.9}$$

(where κ, λ and h are real constants) have been developed by Aceves and Wabnitz (1989) as a generalization of the equations of the classical massive Thirring model (Chang *et al*, 1975). Since pulse-like (bound state) solutions of (3.9) are known, we seek solutions of (2.1) in the form

$$u^\pm(x,t) = f_\pm(t)F_\pm(t)\exp[\mathrm{i}\varphi^\pm(x,t)] \tag{3.10}$$

for suitable real functions $f_\pm(t)$, $\varphi^\pm(x,t)$, with $F_\pm(t)$ satisfying (3.9).

After substituting (3.10) into (2.1) and invoking (3.9), it is possible to show that the system (2.1) admits two cases of the type (3.10). In one case, the coefficients $\alpha_\pm(t)$,

$\beta_\pm(t)$ and $\varepsilon_\pm(t)$ must have the forms

$$\alpha_+ = \frac{t-t_0}{c_+^2(V+1)} \qquad \alpha_- = \frac{h(t-t_0)}{c_+^2(V-1)}$$

$$\beta_+ = \frac{h(t-t_0)}{c_-^2(V+1)} \qquad \beta_- = \frac{t-t_0}{c_-^2(V-1)}$$

$$\varepsilon_\pm = \frac{c_\pm}{c_\mp(V\pm 1)} \exp\left[\pm \mathrm{i}\left\{\frac{2\lambda t}{1-V^2} + 2\chi_0\right\}\right]$$

where $c_\pm$, h, λ, V, t_0 and χ_0 are arbitrary real constants. In this case, equations (2.1) may be reduced by the substitution $u^\pm = c_\pm w^\pm \exp(\pm \mathrm{i}\chi_0)$ to the form

$$(V\pm 1)(\mathrm{i}w_t^\pm - w_{xx}^\pm) = (t-t_0)\{|w^\pm|^2 + h|w^\mp|^2\}w^\pm + w^\mp \exp\left[\pm\mathrm{i}\left(\frac{2\lambda t}{1-V^2}\right)\right] \tag{3.11}$$

while the solutions to (3.10) have the form

$$f_\pm(t) = \frac{c_\pm}{(t-t_0)^{\frac{1}{2}}} \qquad \varphi^\pm(x,t) = \frac{-(x-x_0)^2}{4(t-t_0)} + \frac{\lambda}{V\pm 1}t \pm \chi_0. \tag{3.12}$$

In the second case, the coefficients $\alpha_\pm$, $\beta_\pm$ and $\varepsilon_\pm$ are the same as above, except that the factor $t-t_0$ is removed from $\alpha_\pm$ and $\beta_\pm$. Correspondingly, the substitution

$$u^\pm = c_\pm \exp[\mathrm{i}\{\pm\chi_0 + \lambda t/(V\pm 1)\}v^\pm(x,t)] \tag{3.13}$$

reduces equations (2.1) to the form

$$(V\pm 1)(\mathrm{i}v_t^\pm - v_{xx}^\pm) = (|v^\pm|^2 + h|v^\mp|^2)v^\pm + \lambda v^\pm + v^\mp \tag{3.14}$$

while the corresponding solutions (3.10) have

$$f_\pm(t) = c_\pm \qquad \varphi^\pm(x,t) = mx - \omega t \mp \lambda t(V^2-1)^{-1} \pm \chi_0 \tag{3.15}$$

where ω and m satisfy $\omega + m^2 + \lambda V(V^2-1)^{-1} = 0$.

Since equations (3.9) arise for every travelling wave solution

$$u^\pm(s,\tau) = \mathrm{e}^{-\mathrm{i}\lambda\tau}F_\pm(t) \qquad t = -(s+V\tau)$$

of the system (1.7) with $\delta = 1$, $\kappa = 1$, the special solutions given by Aceves and Wabnitz (1989) in terms of a complex hyperbolic secant describe solutions of (3.11) and (3.14). Solutions to (3.9) are constructed by writing $F_\pm(t) = \hat{F}_\pm(t)\exp[\mathrm{i}(\Phi\mp\Theta)]$, with $\hat{F}_\pm$, Θ and Φ real, and observing that the equation $(V+1)\hat{F}_+\hat{F}'_+(t) + (V-1)\hat{F}_-\hat{F}'_-(t) = 0$ allows solutions with $\hat{F}_\pm \to 0$ simultaneously only if $\hat{F}_+^2 = \beta^4\hat{F}_-^2$, with $\beta \equiv \{(1-V)/(1+V)\}^{\frac{1}{4}}$. Then, by writing $\hat{F}_\pm = \beta^{\pm 1}g(t)$ we find that

$$y'(\xi) = y\sin 2\Theta \qquad \Theta'(\xi) = \Lambda + y^2 + \cos 2\Theta \tag{3.16}$$

where $\xi = (1-V^2)^{-\frac{1}{2}}t = -(s+V\tau)(1-V^2)^{-\frac{1}{2}}$, $g(t) = \gamma y(\xi)$, $\gamma^{-2} = h + (1+V^2)/(1-V^2)$, $\Lambda = \lambda(1-V^2)^{-\frac{1}{2}}$ with $\Phi'(\xi) = V\Lambda + 2Vg^2(1-V^2)^{-1}$.

The system (3.16) has a first integral (Chang *et al* 1975) $y^2(\Lambda + \frac{1}{2}y^2 + \cos 2\Theta) =$ constant, which allows localized (bound-state) solutions only if $y^2 = -2(\Lambda + \cos 2\Theta)$, so $\Phi(\xi) = V\Lambda\xi + 4\gamma^2 V(1-V^2)^{-1}\theta(\xi) + \varphi_0$. Integrating $(3.16)_2$ then gives

$$\tan\Theta = \cot(\tfrac{1}{2}Q)\coth\chi \qquad \chi = (\xi-\xi_0)\sin Q \qquad \cos Q = \Lambda$$

$$\begin{aligned} f(\eta) = \gamma y(\xi) &= \gamma\sin Q\{\cos^2(\tfrac{1}{2}Q)\cosh^2\chi + \sin^2(\tfrac{1}{2}Q)\sinh^2\chi\}^{-\frac{1}{2}} \\ &= \gamma\sin Q|\mathrm{sech}(\chi \pm \mathrm{i}\tfrac{1}{2}Q)| \end{aligned}$$

so that the general bound-state solution to (3.9) is found as

$$F_\pm = C_\pm \sin Q |\operatorname{sech}(\chi + \mathrm{i}\tfrac{1}{2}Q)| \exp \mathrm{i}(\Phi \mp \Theta) = \mp \mathrm{i} C_\pm \sin Q\, \mathrm{e}^{\mathrm{i}\Phi} \operatorname{sech}(\chi \mp \mathrm{i}\tfrac{1}{2}Q) \tag{3.17}$$

where

$$C_\pm = \left\{\frac{1-V}{1+V}\right\}^{\pm\frac{1}{4}} \left\{h + \frac{1+V^2}{1-V^2}\right\}^{-\frac{1}{2}} \qquad \chi = \frac{\sin Q}{\sqrt{1-V^2}}(t - \hat{t}).$$

We conclude that, if a coupled system may be reduced to the constant coefficient form (3.14), solutions exist in which $|F_+|$ and $|F_-|$ depend only on t and are localized near $t = \hat{t}$. We observe that this possibility arises only for $-1 < V < 1$, and that the self-trapping giving rise to the bound-state solution requires counter-propagating signals with anomolous and normal dispersion respectively. The corresponding solutions of (2.1) have

$$|u^\pm| = |c_\pm C_\pm \sin Q \operatorname{sech}(\chi + \tfrac{1}{2}\mathrm{i}Q)|.$$

4. Solutions described by a single NLS equation

The system (1.3) with $\Delta = 0$, $\kappa = 0$ possesses 'linearly polarized' solutions, in which $A^+ = \mathrm{e}^{-2\mathrm{i}\alpha} A^-$ (α a real constant). Similarly, for certain variable-coefficient CNLS systems, reduction to a single variable-coefficient NLS is possible (Ryder and Parker 1992). The aim here is to generalize this feature, by writing in (2.1)

$$u^+(x,t) = p(x,t)\, \mathrm{e}^{\mathrm{i}q(x,t)} u^-(x,t) \tag{4.1}$$

and imposing a simple requirement on the real functions $p(x,t)$ and $q(x,t)$.

By substituting (4.1) into (2.1) and requiring that the coefficients of u_x^-, $|u^-|^2u^-$ and u^- vanish individually from the compatibility condition, we make the deductions $p = p(t)$, $q = q(t)$ with

$$\begin{aligned} &(\alpha_+ - \alpha_-)p^2 + \beta_+ - \beta_- = 0 \\ &pq'(t) + \rho_+ \cos(\sigma_+ - q) - p^2\rho_- \cos(\sigma_- + q) = 0 \\ &p'(t) - \rho_+ \sin(\sigma_+ - q) + p^2\rho_- \sin(\sigma_- + q) = 0 \end{aligned} \tag{4.2}$$

where $\varepsilon_\pm = \rho_\pm \exp\{\mathrm{i}\sigma_\pm(t)\} = \mu_\pm(t) + \mathrm{i}\nu_\pm(t)$. Additionally, u^- *must satisfy the single* NLS equation

$$\mathrm{i}u_t^- = u_{xx}^- + H(t)|u^-|^2u^- + \{S'(t) + \mathrm{i}R'(t)\}u^-$$

in which $H(t) = \alpha_- p^2 + \beta_-$, $S'(t) = p\rho_- \cos(\sigma_- + q)$, $R'(t) = p\rho_- \sin(\sigma_- + q)$. Furthermore, by the transformation

$$u^-(x,t) = \mathrm{e}^{R - \mathrm{i}S} v(x,t) \tag{4.3}$$

it is simple to reduce this to another NLS equation

$$\mathrm{i}v_t = v_{xx} + \hat{H}(t)|v|^2 v \tag{4.4}$$

in which $\hat{H}(t) = H(t)\exp[2R(t)]$.

Consequently, the system (2.1) possesses solutions corresponding to *any* solution of (4.4) whenever the functional form of the eight real coefficients $\alpha_\pm(t)$, $\beta_\pm(t)$, $\rho_\pm(t)$ and $\sigma_\pm(t)$ involved in (2.1) is compatible with the system (4.2). Since these may be rearranged to give

$$p=\left(\frac{\beta_+-\beta_-}{\alpha_--\alpha_+}\right)^{\frac{1}{2}}\equiv p(t) \qquad H(t)=\frac{\alpha_-\beta_+-\alpha_+\beta_-}{\alpha_--\alpha_+}$$
$$\sin[q-\theta(t)]=-P^{-1}p'(t) \tag{4.5}$$

where $\nu_+-p^2\nu_-\equiv P\sin\theta$, $\mu_++p^2\mu_-=P\cos\theta$, the compatibility condition for (4.1) reduces to the equation obtained by substituting $p(t)$ and $q(t)$ from (4.5) into $(4.2)_2$. This provides just one restriction on the coefficients in (2.1).

For (4.4) with general real coefficient $\hat{H}(t)$, several results are known. For example, Joshi (1987) developed the Painlevé analysis for (4.4) obtaining the same constraints as those derived by Grimshaw in connection with the reduction of (4.4) to the constant coefficient case, while Manganaro (1991) classified the functions $\hat{H}(t)$ which allow (4.4) to possess generalized similarity solutions, giving also criteria to characterize compression, amplification or constant amplitude in soliton pulse propagation. Moreover, Grimshaw (1979) has shown, by means of the transformation

$$v=t^{-\frac{1}{2}}\exp\left(\frac{-\mathrm{i}x^2}{4t}\right)w(\tau,\xi) \qquad \xi=x/t \qquad \tau=t^{-1} \tag{4.6}$$

that, whenever $\hat{H}(t)=p_0t^{-1}$ ($p_0=\text{const}$), equation (4.4) reduces to the constant coefficient NLS equation

$$\mathrm{i}w_\tau=w_{\xi\xi}+p_0|w|^2w. \tag{4.7}$$

Since equation (4.7) (cf (1.1)) may be integrated using the inverse scattering method (Zahkarov and Shabat 1972), solutions to a wide class of initial value problems for (2.1) may, in principle, be determined in two cases, (i) $H(t)\,\mathrm{e}^{2R}=\text{constant}$, (ii) $H(t)\,\mathrm{e}^{2R}\propto t^{-1}$. Each of these imposes just one further condition on the coefficients in (2.1), namely

(i) $$\frac{H'(t)}{H(t)}=-2R'(t)=-2p(\nu_-\cos q+\mu_-\sin q)$$

(ii) $$\frac{H'(t)}{H(t)}=-t-2R'(t)=-t-2p(\nu_-\cos q+\mu_-\sin q).$$

The useful possibilities having $\alpha_\pm$ and $\beta_\pm$ constant are readily analysed, since p^2 and H are then constant. Equation (4.5) coupled with case (i) then gives

$$\tan q=\nu_+/\mu_+=-\nu_-/\mu_-=\tan\sigma_+=-\tan\sigma_-$$

so that, in (2.1), we must have $\varepsilon_\pm(t)=\rho_\pm(t)\exp[\pm\mathrm{i}q(t)]$. This gives $\sigma_\pm=\pm q(t)$, so that the only restriction on $\rho_\pm(t)$ and $q(t)$ arises from $(4.2)_2$ and is

$$q'(t)=p\rho_-(t)-p^{-1}\rho_+(t). \tag{4.8}$$

In terms of the coefficients in (2.1) this gives the restriction

$$\frac{\mathrm{d}}{\mathrm{d}t}(\arg\varepsilon_+-\arg\varepsilon_-)=2p|\varepsilon_-|-\frac{2}{p}|\varepsilon_+| \qquad p=\left(\frac{\beta_+-\beta_-}{\alpha_--\alpha_+}\right)^{\frac{1}{2}}$$

while, in terms of the coefficients in (1.12), we find the conditions

$$\arg\tilde{\varepsilon}_++\arg\tilde{\varepsilon}_-=0$$
$$q'(t)=\frac{\mathrm{d}}{\mathrm{d}t}(\arg\tilde{\varepsilon}_+)+\mathrm{Re}(\tilde{\gamma}_+-\tilde{\gamma}_-)=\left(\frac{\tilde{\beta}_+-\tilde{\beta}_-}{\tilde{\alpha}_--\tilde{\alpha}_+}\right)^{\frac{1}{2}}|\tilde{\varepsilon}_-|-\left(\frac{\tilde{\alpha}_--\tilde{\alpha}_+}{\tilde{\beta}_+-\tilde{\beta}_-}\right)^{\frac{1}{2}}|\tilde{\varepsilon}_+|. \tag{4.9}$$

For the system (1.3), equation $(4.9)_1$ is satisfied while $(4.9)_2$ gives $q'(t)=2\Delta=0$. Thus, in the cases (i) $\Delta=0$, $h=0$ and (ii) $\Delta=0$ (which was shown in section 1 to correspond to equations (1.4) and (1.5) also) the system (1.3) possesses classes of solutions $A^-=\pm A^+$ which are derivable from solutions of a single NLS equation. Moreover, it is shown in the appendix that the system (1.6) may be transformed to (1.12) with $\tilde{\gamma}_\pm=0$, $\tilde{\varepsilon}_\pm=0$, so allowing classes of solutions of the form (4.1).

Appendix

Here we describe point transformations which link systems (1.3)–(1.6) to the variable coefficient systems of the class (1.12).

First, we observe that (1.6) can be converted to the form (1.12) in two essentially distinct manners:

$$\text{(i)}\qquad \begin{gathered} u^\pm(\tau,s)=\hat{\delta}_\pm(\tau)\exp[\mathrm{i}\{-\Lambda^\pm s-\Lambda^+\Lambda^-\tau+R_\pm(\tau)\}]\tilde{u}^\pm(t,x) \\ x=s+2\lambda\tau \qquad t=-\tau \qquad \tilde{\gamma}_\pm=-R'_\pm(\tau)\pm\Delta+\mathrm{i}\hat{\delta}'_\pm/\hat{\delta}_\pm \end{gathered} \tag{A.1}$$

which yields (1.12) with

$$\tilde{\alpha}_+=\hat{\delta}^2_+ \qquad \tilde{\beta}_-=\hat{\delta}^2_- \qquad \tilde{\alpha}_-=h\tilde{\alpha}_+ \qquad \tilde{\beta}_+=h\tilde{\beta}_- \qquad \tilde{\varepsilon}_\pm=0 \tag{A.2}$$

while $\hat{\delta}_\pm(\tau)$, Re $\tilde{\gamma}_\pm(\tau)$ and the constant λ are arbitrary, with $\Lambda^\pm=\lambda\pm\frac{1}{2}\delta$.

$$\text{(ii)}\qquad \begin{gathered} u^\pm(\tau,s)=f_\pm(\tau)\exp\left[\mathrm{i}\left\{\frac{s^2}{4\tau}\mp\tfrac{1}{2}\delta s+\tfrac{1}{4}\delta^2\tau+R_\pm(\tau)\right\}\right]\tilde{u}^\pm(t,x) \\ x=s/\tau \qquad t=\tau^{-1} \qquad \tilde{\gamma}_\pm=\left(-R'_\pm(\tau)\pm\Delta+\mathrm{i}\frac{f'_\pm}{f_\pm}\right)\tau^2+\frac{\mathrm{i}\tau}{2} \end{gathered} \tag{A.3}$$

which transforms (1.6) to (1.12) with

$$\tilde{\alpha}_+=t^2f^2_+ \qquad \tilde{\beta}_-=t^2f^2_- \qquad \tilde{\alpha}_-=h\tilde{\alpha}_+ \qquad \tilde{\beta}_+=h\tilde{\beta}_- \qquad \tilde{\varepsilon}_\pm=0. \tag{A.4}$$

In case (i), the choices $\hat{\delta}_\pm=1$, $R_\pm=\pm\Delta\tau$ give

$$\mathrm{i}\tilde{u}^\pm_t=\tilde{u}^\pm_{xx}+(|\tilde{u}^\pm|^2+h|\tilde{u}^\mp|^2)\tilde{u}^\pm \tag{A.5}$$

which is equivalent to the special case $\Delta=0$, $\kappa=0$ of (1.3) or $\kappa_\pm=0$, $\mu=0$, $\bar{\varepsilon}_\pm=0$ of (2.12). Equations (A.5) are known to possess many similarity reductions (Parker and Newboult, 1989). Moreover, the special cases with $\tilde{u}^+=\tilde{u}^-\exp(\mathrm{i}\varphi_0)$ give rise to the explicit solutions of Tratnik (1992). In case (ii), the choices $f_\pm=\tau^{-\frac{1}{2}}$, $R_\pm=\pm\Delta\tau$ give

$$\mathrm{i}\tilde{u}^\pm_t=\tilde{u}^\pm_{xx}+t^3(|\tilde{u}^\pm|^2+h|\tilde{u}^\mp|^2)\tilde{u}^\pm \tag{A.6}$$

while the choices $f_\pm=\tau$, $R_\pm=\pm\Delta\tau$ relate to (2.14) with $\kappa_\pm=\bar{\varepsilon}_\pm=D=0$.

Consequently, since the coefficients in (A.5) agree with both the forms (2.7) and (2.8), the system (1.6) possesses exact similarity solutions derived from Cases 1–3 for (A.5). Moreover, the coefficients in (A.6) also agree with (2.8) for $\kappa_\pm=\frac{7}{4}$, $\bar{\varepsilon}_\pm=0$, so allowing solutions derived from Case 3 for (A.6).

Similar results apply for the system (1.3), which is transformed to the form (1.12) in the two cases:

$$\text{(iii)} \qquad \begin{aligned} &A^{\pm}(\tau, s) = \delta_{\pm}(\tau) \exp[\mathrm{i}\{-\lambda(s+\lambda\tau)+R_{\pm}(\tau)\}]\tilde{u}^{\pm}(t, x) \\ &x = s+2\lambda\tau \qquad t=-\tau \qquad \tilde{\gamma}_{\pm} = -R'_{\pm}(\tau) \pm \Delta + \mathrm{i}(\delta'_{\pm}/\delta_{\pm}) \end{aligned} \tag{A.7}$$

which yields (1.12) with

$$\begin{aligned} &\tilde{\alpha}_{+} = \delta_{+}^{2} \qquad \tilde{\beta}_{-} = \delta_{-}^{2} \qquad \tilde{\alpha}_{-} = h\tilde{\alpha}_{+} \qquad \tilde{\beta}_{+} = h\tilde{\beta}_{-} \\ &\tilde{\varepsilon}_{\pm} = \kappa \frac{\delta_{\mp}}{\delta_{\pm}} \exp[\mp \mathrm{i}(R_{+}-R_{-})] \end{aligned} \tag{A.8}$$

$$\text{(iv)} \qquad \begin{aligned} &A^{\pm}(\tau, s) = f_{\pm}(\tau) \exp\left[\mathrm{i}\left\{\frac{s^{2}}{4\tau}+R_{\pm}(\tau)\right\}\right]\tilde{u}^{\pm}(t, x) \\ &x = s/\tau \qquad t = \tau^{-1} \qquad \tilde{\gamma}_{\pm} = \left(-R'_{\pm}(\tau) \pm \Delta + \mathrm{i}\frac{f'_{\pm}}{f_{\pm}}\right)\tau^{2} + \mathrm{i}\frac{\tau}{2} \end{aligned} \tag{A.9}$$

which transforms (1.3) to (1.12) with

$$\begin{aligned} &\tilde{\alpha}_{+} = t^{2} f_{+}^{2} \qquad \tilde{\beta}_{-} = t^{2} f_{-}^{2} \qquad \tilde{\alpha}_{-} = h\tilde{\alpha}_{+} \qquad \tilde{\beta}_{+} = h\tilde{\beta}_{-} \\ &\tilde{\varepsilon}_{\pm} = \kappa \frac{f_{\mp}}{f_{\pm}} \tau^{2} \exp \mp \mathrm{i}(R_{+}-R_{-}). \end{aligned} \tag{A.10}$$

Also, in these cases it is readily shown that, for $\Delta = 0$, the system (1.3) yields $\kappa_{\pm} = 0$ in (2.12) in Case (iii), and yields $\kappa_{\pm} = D = 0$ in (2.14) in Case (iv).

Since the systems (1.4) and (1.5) can each be transformed to (1.3) with $\Delta = 0$, they also may each be related to (1.12) by either (A.7) or (A.9).

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